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Smooth continuation of solutions and eigenvalues for variational inequalities based on the implicit function theorem

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Abstract

The implicit function theorem is applied in a nonstandard way to abstract variational inequalities depending on a (possibly infinite-dimensional) parameter. In this way, results on smooth continuation of solutions as well as of eigenvalues and eigenvectors are established under certain particular assumptions. The abstract results are applied to a linear second order elliptic eigenvalue problem with nonlocal unilateral boundary conditions (Schrödinger operator with the potential as the parameter).

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1. Introduction

This paper concerns smooth continuation of solutions to parameter depending variational inequalities of the type

$$\lambda \in \Lambda,\ u \in K:\quad \langle u - F(\lambda, u), \varphi - u\rangle \geqslant 0 \quad \text{for all } \varphi \in K \tag{1.1}$$

and of eigenvalues and eigenvectors satisfying parameter depending eigenvalue problems of the type

$$\lambda \in \Lambda,\ \mu \in \mathbb{R},\ u \in K:\quad \langle \mu u - L(\lambda)u, \varphi - u\rangle \geqslant 0 \quad \text{for all } \varphi \in K. \tag{1.2}$$

Here K is a closed convex cone in a real Hilbert space H, $\lambda \in \Lambda$ is the "control" parameter, with respect to which continuation takes place, and Λ is a normed vector space. In (1.1) and (1.2), $F : \Lambda \times H \to H$ and $L : \Lambda \to \mathcal{L}(H)$ are smooth maps, respectively.

Our aim is to apply the implicit function theorem to (1.1) and (1.2) and, hence, to construct smooth solution families $u = \hat{u}(\lambda)$ to (1.1) and $\mu = \hat{\mu}(\lambda)$, $u = \hat{u}(\lambda)$ to (1.2), respectively. The basic idea is to show that, near a given solution (λ_0, u_0), solutions to our variational inequality coincide with those of an equation $u = PF(\lambda, u)$, where P is the projection onto a suitable subspace H_0 (see below). The implicit function theorem is applied in fact to this equation, which is, in contrast to the original variational inequality, smooth and can be linearized. In particular, we will have $\hat{u}(\lambda) \in \partial K$ for all λ considered.

Let us remark that our results are purely local. In particular, we do not use any assumptions concerning the "global" behavior of the map F like monotonicity, coercivity or growth conditions. All our assumptions concern only the value $F(\lambda_0, u_0)$ and the linear operator $(\partial F/\partial u)(\lambda_0, u_0)$.

The results for the inequality (1.2) are transformed also to a more general problem such that the weak form of the following example is included:

$$\Delta u(x) + \lambda(x)u(x) = \mu u(x) \quad \text{in } \Omega, \tag{1.3}$$

$$u = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_U), \tag{1.4}$$

$$\int_{\Gamma_U} u \,\mathrm{d}\Gamma \geqslant 0, \qquad \frac{\partial u}{\partial n} \geqslant 0, \qquad \frac{\partial u}{\partial n} \text{ is constant,}$$

$$\int_{\Gamma_U} u \,\mathrm{d}\Gamma \cdot \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_U. \tag{1.5}$$

Here Ω is a bounded domain in $\mathbb{R}^n$, Γ_D and Γ_U are subsets of the boundary $\partial\Omega$, $\lambda \in L^\infty(\Omega)$ is the control parameter, $\mu \in \mathbb{R}$ is the eigenvalue parameter. For this particular example, our idea is very simple. We consider a given λ_0 such that there is an eigenvalue μ_0 having a unique normed eigenvector u_0 satisfying $\partial u_0/\partial n > 0$ on Γ_U (which implies $\int_{\Gamma_U} u_0 \,\mathrm{d}\Gamma = 0$). This property is preserved for all possible solutions μ, λ, u in a neighbourhood of μ_0, λ_0, u_0. In particular, such solutions satisfy (1.3), (1.4) and

$$\int_{\Gamma_U} u \,\mathrm{d}\Gamma = 0, \qquad \frac{\partial u}{\partial n} \text{ is constant on } \Gamma_U. \tag{1.6}$$

Moreover, we can show that in this neighbourhood, the problem (1.3), (1.4), (1.5) (which is nonsmooth and cannot be linearized in general) is equivalent to (1.3), (1.4), (1.6) (which is smooth and has a natural linearization). Now, implicit function theorem for the weak form of (1.3), (1.4), (1.6) can be used and a smooth family $\mu = \hat{\mu}(\lambda)$, $u = \hat{u}(\lambda)$ of solutions to (1.3), (1.4), (1.6) (together with a normalization condition) is obtained. These solutions are simultaneously the only solutions to (1.3), (1.4), (1.5) (together with a normalization condition) on the neighbourhood of μ_0, λ_0, u_0.

In general, we consider the particular case when the cone K is the intersection of half spaces:

$$K = \bigl\{u \in H\colon \langle u, v_\alpha\rangle \geqslant 0 \text{ for all } \alpha \in \mathcal{A}\bigr\}.$$

Here $\mathcal{A}$ is a set and $\{v_\alpha\}_{\alpha\in\mathcal{A}}$ is a family of vectors in H. Moreover, we deal with cases when there exists a nonempty subset $\mathcal{A}_0$ in $\mathcal{A}$ such that for all λ under consideration we have

$$\begin{aligned} &\bigl\langle \hat{u}(\lambda), v_\alpha\bigr\rangle = 0 \quad \text{for all } \alpha \in \mathcal{A}_0,\\ &\bigl\langle \hat{u}(\lambda), v_\alpha\bigr\rangle > 0 \quad \text{for all } \alpha \in \mathcal{A}\setminus\mathcal{A}_0. \end{aligned} \tag{1.7}$$

In applications this means that the set of the “active” constraints $\mathcal{A}_0$ does not depend on the control parameter λ. In a forthcoming paper we will study also cases when the set of the “active” constraints of a family of smooth solutions $\hat{u}(\lambda)$ depends, in a suitably defined smooth way, on λ. Let us remark that our existence results produce solutions satisfying (1.7), but our results on local uniqueness hold true in the whole cone K.

Our paper is organized as follows.

In Section 3 we state sufficient conditions under which the variational inequality (1.1) near (λ_0, u_0) can be uniquely solved in terms of $u = \hat{u}(\lambda)$, where the map $\hat{u}$ is smooth. Some former results of other authors concerning a continuation for variational inequalities are briefly mentioned at the end of Section 3. However, as far as we know, nothing is known about smoothness of such solution families obtained.

In Section 4 we prove unique smooth continuation of simple eigenvalues μ for inequalities of the type (1.2). This result can be understood as a certain analogue of [2, Theorem 14.3.1] (in a particular Hilbert space setting) for variational inequalities. Further, we prove an analogous result for variational inequalities of a more general type than (1.2).

In Section 5 we apply the abstract results of Section 4 to a linear elliptic eigenvalue problem with mixed boundary conditions, including unilateral integral boundary conditions (the problem (1.3)–(1.5) and its generalization). Note that the data of this problem can be nonsmooth (L^∞-coefficients, Lipschitz boundary).

Applications of the results of Section 3 to a boundary value problem for a nonlinear fourth order ordinary differential equation with pointwise unilateral conditions (obstacle problem for a beam equation) will be given in [11], which is in fact a continuation of the present paper. In [11], abstract variational inequalities of the type (1.1) with $F(\lambda, 0) = 0$ are considered and smooth branches of solutions, bifurcating from the trivial solution, are obtained. This result can be understood as a certain analogue of the so-called "main theorem on generic bifurcation for multiparameter operator equations (bunch theorem)" [13, Chapter 8.11] (in a particular Hilbert space setting) for variational inequalities.

The results of the present paper and of [11] are presented already in the preprint [5].

2. Notation, setting and some lemmas

In Sections 2–4 of this paper H is a real Hilbert space with a scalar product $\langle\cdot\,,\cdot\rangle$ and a norm $\|\cdot\|$, $\mathcal{A}$ is a nonempty set and $\{v_\alpha\}_{\alpha\in\mathcal{A}} \subset H$ is a family of vectors in H such that

$$\|v_\alpha\| = 1 \quad \text{for all } \alpha \in \mathcal{A}. \tag{2.1}$$

By K we denote the closed convex cone in H which is defined by

$$K := \big\{u \in H\colon \langle u, v_\alpha\rangle \geqslant 0 \text{ for all } \alpha \in \mathcal{A}\big\},$$

and $P_K : H \to K$ is the projection of H onto K. As is well known (cf., e.g., [6, Section 1.2] or [1, Chapter 3]), for $u \in H$ the element $P_K(u)$ is uniquely defined by the condition

$$P_K(u) \in K \quad \text{and} \quad \big\|u - P_K(u)\big\| \leqslant \|u - \varphi\| \quad \text{for all } \varphi \in K \tag{2.2}$$

or by the condition

$$P_K(u) \in K \quad \text{and} \quad \big\langle P_K(u) - u, \varphi - P_K(u)\big\rangle \geqslant 0 \quad \text{for all } \varphi \in K. \tag{2.3}$$

Further, $\mathcal{A}_0$ is a fixed subset of $\mathcal{A}$,

$$H_0 := \big\{u \in H\colon \langle u, v_\alpha\rangle = 0 \text{ for all } \alpha \in \mathcal{A}_0\big\}$$

is the corresponding closed subspace in H, and we denote by $P \in \mathcal{L}(H)$ the orthogonal projection from H onto H_0. Finally, Λ is a normed vector space (the norm of which will be denoted by $\|\cdot\|$, too) and $F : \Lambda \times H \to H$ a continuous map.

Lemma 2.1. *Let $\lambda_0 \in \Lambda$, $u_0 \in H$ and $c > 0$ be such that $u_0 = PF(\lambda_0, u_0)$ and*

$$\langle u_0, v_\alpha\rangle \geqslant c \quad \textit{for all } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{2.4}$$

$$\big\langle F(\lambda_0, u_0), (I - P)\varphi\big\rangle \leqslant -c\big\|(I - P)\varphi\big\| \quad \textit{for all } \varphi \in K. \tag{2.5}$$

Then there exists an $\varepsilon > 0$ *such that* $P_K(F(\lambda, u)) = PF(\lambda, u)$ *holds for all* $\lambda \in \Lambda$ *and* $u \in H$ *with* $\|\lambda - \lambda_0\| + \|u - u_0\| < \varepsilon$.

Proof. Because of the characterization (2.3) of the projection P_K we have to show that $PF(\lambda, u) \in K$ and $\langle PF(\lambda, u) - F(\lambda, u), \varphi - PF(\lambda, u)\rangle \geqslant 0$ for all $\lambda \approx \lambda_0$, $u \approx u_0$ and $\varphi \in K$, i.e., that

$$\big\langle PF(\lambda, u), v_\alpha \big\rangle \geqslant 0 \quad \text{for all } \lambda \approx \lambda_0,\ u \approx u_0 \text{ and } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{2.6}$$

$$\big\langle (I - P)F(\lambda, u), \varphi \big\rangle \leqslant 0 \quad \text{for all } \lambda \approx \lambda_0,\ u \approx u_0 \text{ and } \varphi \in K. \tag{2.7}$$

Suppose that (2.6) is not true. Then there exist sequences $(\lambda_j) \subset \Lambda$, $(u_j) \subset H$ and $(\alpha_j) \subset \mathcal{A} \setminus \mathcal{A}_0$ such that $\lambda_j \to \lambda_0$ and $u_j \to u_0$ for $j \to \infty$ and $\langle PF(\lambda_j, u_j), v_{\alpha_j}\rangle < 0$. Because of (2.1) and (2.4) this implies

$$\begin{aligned} 0 &> \langle u_0, v_{\alpha_j}\rangle + \big\langle P\big(F(\lambda_j, u_j) - F(\lambda_0, u_0)\big), v_{\alpha_j}\big\rangle \\ &\geqslant c - \big\|F(\lambda_j, u_j) - F(\lambda_0, u_0)\big\|, \end{aligned}$$

which is, for large j, a contradiction.

Now, suppose that (2.7) is not true. Then there exist sequences $(\lambda_j) \subset \Lambda$, $(u_j) \subset H$ and $(\varphi_j) \subset K$ such that $\lambda_j \to \lambda_0$ and $u_j \to u_0$ for $j \to \infty$ and $\langle (I - P)F(\lambda_j, u_j), \varphi_j\rangle > 0$. Because of (2.5) this implies

$$\begin{aligned} 0 &< \big\langle F(\lambda_0, u_0), (I - P)\varphi_j\big\rangle + \big\langle F(\lambda_j, u_j) - F(\lambda_0, u_0), (I - P)\varphi_j\big\rangle \\ &\leqslant \big(-c + \big\|F(\lambda_j, u_j) - F(\lambda_0, u_0)\big\|\big)\big\|(I - P)\varphi_j\big\|, \end{aligned}$$

which is, for large j, a contradiction. □

Lemma 2.2. *Let* $\lambda_0 \in \Lambda$, $u_0 \in H_0$ *and* $c > 0$ *be such that* $u_0 = P_K(F(\lambda_0, u_0))$ *and that* (2.4) *holds. Then* $P_K(F(\lambda_0, u_0)) = PF(\lambda_0, u_0)$. *If, moreover, the condition* (2.5) *is fulfilled then there exists an* $\varepsilon > 0$ *such that* $P_K(F(\lambda, u)) = PF(\lambda, u)$ *holds for all* $\lambda \in \Lambda$ *and* $u \in H$ *with* $\|\lambda - \lambda_0\| + \|u - u_0\| < \varepsilon$.

Proof. Denote $w := F(\lambda_0, u_0)$. Suppose that $P_K(w) \neq Pw$. We have $P_K(w) = u_0 \in H_0$ and therefore (cf. (2.2))

$$\|Pw - w\| = \min_{v \in H_0} \|v - w\| < \big\|P_K(w) - w\big\| = \min_{v \in K} \|v - w\|.$$

Because of (2.4) there is $\tau \in (0, 1)$ such that $v = \tau Pw + (1 - \tau)P_K(w)$ satisfies $\langle v, v_\alpha\rangle \geqslant 0$ for all $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$. We get $v \in H_0 \cap K$ and simultaneously

$$\|v - w\| \leqslant \tau\|Pw - w\| + (1 - \tau)\big\|P_K(w) - w\big\| < \big\|P_K(w) - w\big\|,$$

which is a contradiction and the first assertion is proved. The second assertion follows now from Lemma 2.1. □

Lemma 2.3. *Suppose that $\mathcal{A}_0$ is a finite set and assume that the vectors $\{v_\alpha\}_{\alpha\in\mathcal{A}_0}$ are linearly independent. Then there exists a basis $\{v_\alpha^*\}_{\alpha\in\mathcal{A}_0}$ in* $\operatorname{span}\{v_\alpha\colon \alpha\in\mathcal{A}_0\}$ *such that $\langle v_\alpha^*, v_\beta\rangle = \delta_{\alpha\beta}$ for $\alpha,\beta\in\mathcal{A}_0$ and*

$$(I-P)u = \sum_{\alpha\in\mathcal{A}_0}\bigl\langle u, v_\alpha^*\bigr\rangle v_\alpha = \sum_{\alpha\in\mathcal{A}_0}\langle u, v_\alpha\rangle v_\alpha^* \quad \text{for all } u\in H. \tag{2.8}$$

If, moreover, $\lambda_0\in\Lambda$ and $u_0\in H_0$ satisfies (2.4) *then the condition* (2.5) *is equivalent to*

$$\bigl\langle F(\lambda_0,u_0), v_\alpha^*\bigr\rangle < 0 \quad \text{for all } \alpha\in\mathcal{A}_0. \tag{2.9}$$

Proof. The existence of the dual basis $\{v_\alpha^*\}_{\alpha\in\mathcal{A}_0}$ in $\operatorname{span}\{v_\alpha\colon \alpha\in\mathcal{A}_0\}$ and the structure (2.8) of the orthoprojector onto $\operatorname{span}\{v_\alpha\colon \alpha\in\mathcal{A}_0\}$ are well-known.

Let $\alpha\in\mathcal{A}_0$ and let $\varepsilon\geqslant 0$ be small. Then $u_0+\varepsilon v_\alpha^*\in K$, and (2.5) yields

$$\begin{aligned}\bigl\langle F(\lambda_0,u_0), (I-P)\bigl(u_0+\varepsilon v_\alpha^*\bigr)\bigr\rangle &= \varepsilon\bigl\langle F(\lambda_0,u_0), (I-P)v_\alpha^*\bigr\rangle\\ &\leqslant -c\varepsilon\bigl\|(I-P)v_\alpha^*\bigr\|.\end{aligned}$$

Simultaneously $(I-P)v_\alpha^* = v_\alpha^*$ by (2.8). Hence, (2.5) implies (2.9).

If (2.5) is not satisfied then there exists a sequence $(\varphi_j)\subset K$ such that $\|(I-P)\varphi_j\|\neq 0$ and

$$\left\langle F(\lambda_0,u_0), \frac{(I-P)\varphi_j}{\|(I-P)\varphi_j\|}\right\rangle > -\frac{1}{j} \tag{2.10}$$

for all j. We can assume that

$$\frac{(I-P)\varphi_j}{\|(I-P)\varphi_j\|}\to\varphi_0 = (I-P)\varphi_0 \quad \text{for } j\to\infty.$$

Hence, we have $\varphi_0 = \sum_{\alpha\in\mathcal{A}_0}\langle\varphi_0, v_\alpha\rangle v_\alpha^*$, and (2.10) yields

$$\bigl\langle F(\lambda_0,u_0),\varphi_0\bigr\rangle = \sum_{\alpha\in\mathcal{A}_0}\bigl\langle F(\lambda_0,u_0), v_\alpha^*\bigr\rangle\langle\varphi_0, v_\alpha\rangle \geqslant 0.$$

Moreover, since $\varphi_j\in K$ we have for all $\alpha\in\mathcal{A}_0$ that

$$\langle\varphi_0, v_\alpha\rangle = \lim_{j\to\infty}\left\langle\frac{(I-P)\varphi_j}{\|(I-P)\varphi_j\|}, v_\alpha\right\rangle = \lim_{j\to\infty}\left\langle\frac{\varphi_j}{\|(I-P)\varphi_j\|}, v_\alpha\right\rangle \geqslant 0.$$

Finally, because of $\|\varphi_0\| = 1$ we have that $\langle\varphi_0, v_\alpha\rangle > 0$ for at least one $\alpha\in\mathcal{A}_0$. Hence, (2.9) cannot be fulfilled. □

Remark 2.1. If $u_0 = PF(\lambda_0,u_0)$ then

$$\bigl\langle u_0 - F(\lambda_0,u_0), \varphi - u_0\bigr\rangle = -\bigl\langle F(\lambda_0,u_0), (I-P)\varphi\bigr\rangle \quad \text{for all } \varphi\in H. \tag{2.11}$$

Hence, in Lemma 2.1 and in Lemma 2.2, the condition (2.5) can be replaced by

$$\big\langle u_0 - F(\lambda_0, u_0), \varphi - u_0\big\rangle \geqslant c\big\|(I-P)\varphi\big\| \quad \text{for all } \varphi \in K. \tag{2.12}$$

In particular, if the assumptions of Lemma 2.2 are satisfied with the exception of (2.5) then we already have $\langle F(\lambda_0, u_0), (I-P)\varphi\rangle \leqslant 0$ for all $\varphi \in K$ because of (1.1) and (2.11). Hence, the role of the assumption (2.5) is only to ensure that this inequality is in a certain sense uniform and strict. Moreover, $(I-P)\varphi \neq 0$ for $\varphi \in K$ if and only if $\langle \varphi, v_\alpha\rangle > 0$ for a certain $\alpha \in \mathcal{A}_0$. Thus, if $\mathcal{A}_0$ is finite then both (2.12) and (2.5) are equivalent to

$$\begin{aligned}&\big\langle u_0 - F(\lambda_0, u_0), \varphi - u_0\big\rangle > 0 \quad \text{for all } \varphi \in K\\ &\qquad \text{such that } \langle \varphi, v_\alpha\rangle > 0 \text{ for a certain } \alpha \in \mathcal{A}_0.\end{aligned} \tag{2.13}$$

This equivalence follows from the fact that (2.12) implies (2.13) and (2.13) with the help of (2.11) and of the considerations from the proof of Lemma 2.3 implies (2.5), which is equivalent to (2.12).

Remark 2.2. In many applications the variational inequality

$$\lambda \in \Lambda,\ u \in K\colon \quad \big\langle u - F(\lambda, u), \varphi - u\big\rangle \geqslant 0 \quad \text{for all } \varphi \in K$$

can be interpreted as a principle of virtual work: Here $F(\lambda, u) - u$ is the generalized force, and the admissible state $u \in K$ is an equilibrium if the work, carried out by the generalized force on arbitrary admissible virtual displacements $\varphi - u$ with $\varphi \in K$, is nonpositive. Using this language, condition (2.13) can be interpreted in the following way: If there is a nonactive constraint for a given admissible virtual displacement among those active for u_0, then the work, carried out by the generalized force on such displacement, should be negative.

Remark 2.3. Let us consider the special case $\mathcal{A} = \mathcal{A}_0 = \{1\}$. Then $v_1^* = v_1$ and $H_0 \subset K$. If (λ, u) satisfies (1.1), i.e.,

$$u = P_K\big(F(\lambda, u)\big), \tag{2.14}$$

then (2.9) is fulfilled if and only if (λ, u) does not satisfy the equation

$$u = F(\lambda, u). \tag{2.15}$$

Indeed, if (2.9) holds then $F(\lambda, u) \notin K$, i.e., $F(\lambda, u) \neq P_K(F(\lambda, u))$, and (2.15) cannot hold because of (2.14). If (2.9) is not true then $F(\lambda, u) \in K$ and this together with (2.14) gives $u - F(\lambda, u) = u - P_K(F(\lambda, u)) = 0$, i.e., (2.15) holds.

3. Unique smooth continuation of solutions

In this section we consider the parameter depending variational inequality

$$\lambda \in \Lambda,\ u \in K\colon \quad \big\langle u - F(\lambda, u), \varphi - u\big\rangle \geqslant 0 \quad \text{for all } \varphi \in K. \tag{3.1}$$

Here $F : \Lambda \times H \to H$ is a C^k-map with $k \geqslant 1$.

Theorem 3.1. *Let a solution* (λ_0, u_0) *to* (3.1) *and* $c > 0$ *be such that*

$$\langle u_0, v_\alpha \rangle = 0 \quad \textit{for all } \alpha \in \mathcal{A}_0, \tag{3.2}$$

$$\langle u_0, v_\alpha \rangle \geqslant c \quad \textit{for all } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{3.3}$$

$$\bigl\langle F(\lambda_0, u_0), (I - P)\varphi \bigr\rangle \leqslant -c \bigl\| (I - P)\varphi \bigr\| \quad \textit{for all } \varphi \in K, \tag{3.4}$$

and that

$$u \in H_0 \mapsto u - P \frac{\partial F}{\partial u}(\lambda_0, u_0) u \in H_0 \quad \textit{is bijective.} \tag{3.5}$$

Then there exist neighbourhoods $U \subseteq H$ *of* u_0 *and* $V \subseteq \Lambda$ *of* λ_0 *and a* C^k*-map* $\hat{u} : V \to H_0$ *such that* $(\lambda, u) \in V \times U$ *satisfies* (3.1) *if and only if* $u = \hat{u}(\lambda)$*. In particular,* $u_0 = \hat{u}(\lambda_0)$*.*

Proof. Let us recall that the inequality (3.1) is equivalent to the equation

$$u - P_K\bigl(F(\lambda, u)\bigr) = 0. \tag{3.6}$$

Because of the assumptions (3.2)–(3.4) and Lemma 2.2 we have $PF(\lambda, u) = P_K(F(\lambda, u))$ for all $\lambda \in \Lambda$ close to λ_0 and $u \in H$ close to u_0. Thus, we have to solve the equation

$$u - PF(\lambda, u) = 0 \tag{3.7}$$

for all $\lambda \in \Lambda$ close to λ_0 and $u \in H$ close to u_0. The linearization of the left hand side of (3.7) in the solution (λ_0, u_0) is $u \mapsto u - P(\partial F/\partial u)(\lambda_0, u_0)u$. This is an isomorphism from H_0 onto H_0 because of the assumption (3.5). Hence, (3.7) can be locally solved in H_0 by the implicit function theorem in terms of $u = \hat{u}(\lambda)$. We have $u \in H_0$ for any solution of (3.7) and therefore the uniqueness assertion of Theorem 3.1 follows from the equivalence of (3.6) and (3.7) and from the uniqueness assertion of the implicit function theorem. □

Remark 3.1. If the operator $(\partial F/\partial u)(\lambda_0, u_0)$ is compact, then the map in condition (3.5) is bijective if and only if it is injective; i.e., in that case (3.5) is satisfied if and only if there is no nontrivial solution to $u = P(\partial F/\partial u)(\lambda_0, u_0)u$.

There exist various results concerning continuation (local existence and uniqueness and continuous dependence on data) of solutions to variational inequalities.

A class of elliptic variational inequalities is studied by Conrad et al. in [3] using conical derivatives of the nonlinear map in an equation equivalent to the variational inequality. General abstract variational inequalities with potential operators are considered (also from the numerical point of view) by Miersemann and Mittelmann (see [7,8] and references therein). The control parameter λ is real and enters linearly in the variational inequalities. On the other hand, general closed convex sets K are considered in the results mentioned. Domokos in [4] works with a generalized implicit function theorem for inequalities with monotone operators. In [12], Yen and Lee consider abstract variational inequalities (where also K can depend on λ) such that the solution depends Hölder continuously on λ. A large class of quasi-variational inclusions is treated by M.A. Noor and K.I. Noor in [10] using the equivalence of that inclusions to so-called parameter depending resolvent equations (for other applications of the resolvent equations technique by M.A. Noor see references therein). M.A. Noor et al. give an comprehensive review of modern trends and achievements in variational and quasi-variational inequalities in [9], including local uniqueness of solutions and their Lipschitz continuous dependence on parameters.

4. Unique smooth continuation of simple eigenvalues

In this section we consider the parameter depending eigenvalue problem

$$\lambda \in \Lambda,\ \mu \in \mathbb{R},\ u \in K\colon \quad \big\langle \mu u - L(\lambda)u, \varphi - u\big\rangle \geqslant 0 \\ \text{for all } \varphi \in K. \tag{4.1}$$

Here $L : \Lambda \to \mathcal{L}(H)$ is a C^k-map with $k \geqslant 1$.

Theorem 4.1. *Let a solution* (λ_0, μ_0, u_0) *to* (4.1) *and* $c > 0$ *be such that* $\mu_0 > 0$ *and*

$$\langle u_0, v_\alpha\rangle = 0 \quad \textit{for all } \alpha \in \mathcal{A}_0, \tag{4.2}$$

$$\langle u_0, v_\alpha\rangle \geqslant c \quad \textit{for all } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{4.3}$$

$$\big\langle L(\lambda_0)u_0, (I - P)\varphi\big\rangle \leqslant -c\big\|(I - P)\varphi\big\| \quad \textit{for all } \varphi \in K, \tag{4.4}$$

$$\dim \ker\big(\mu_0 I - PL(\lambda_0)\big) = 1, \tag{4.5}$$

$$u_0 \notin \big(\mu_0 I - PL(\lambda_0)\big)H_0, \tag{4.6}$$

and that the following simplicity condition is satisfied:

$$\textit{if } (\lambda_0, \mu_0, u) \textit{ is a solution to } (4.1) \textit{ with } \|u\| = \|u_0\|, \\ \textit{then} \quad u = u_0. \tag{4.7}$$

Further, let $L(\lambda_0)$ be compact.

Then there exist neighbourhoods $V \subseteq \Lambda$ of λ_0 and $W \subseteq \mathbb{R}$ of μ_0 and C^k-maps $\hat{\mu} : V \to \mathbb{R}$ and $\hat{u} : V \to H_0$ such that $(\lambda, \mu, u) \in V \times W \times H$ satisfies (4.1) *with $\|u\| = \|u_0\|$ if and only if $\mu = \hat{\mu}(\lambda)$ and $u = \hat{u}(\lambda)$. In particular, $\mu_0 = \hat{\mu}(\lambda_0)$ and $u_0 = \hat{u}(\lambda_0)$.*

Proof. The problem (4.1) with $\mu > 0$ is equivalent to the equation $\mu u = P_K(L(\lambda)u)$. Because of the assumptions (4.2)–(4.4) and of Lemma 2.2 used for the mapping $F(\lambda,u) := (1/\mu_0)L(\lambda)u$ we have $P_K(L(\lambda)u) = PL(\lambda)u$ for all $\lambda \in \Lambda$ close to λ_0 and for all $u \in H$ close to u_0. In particular, $u_0 \in H_0$. Thus, the problem (4.1) with $\lambda \approx \lambda_0$ and $u \approx u_0$ is equivalent to

$$\mu u = PL(\lambda)u. \tag{4.8}$$

In particular, $u_0 \in \ker(\mu_0 I - PL(\lambda_0))$. Since $L(\lambda_0)$ is compact and the assumptions (4.5) and (4.6) are satisfied we have that μ_0 is an (algebraically) simple eigenvalue (in the sense of [2, Section 14.3]) of the operator $u \in H_0 \mapsto PL(\lambda_0)u \in H_0$. Hence, [2, Theorem 14.3.1] works and yields neighbourhoods $V \subseteq \Lambda$ of λ_0 and $W \subseteq \mathbb{R}$ of μ_0 and C^k-maps $\hat{\mu}: V \to \mathbb{R}$ and $\hat{u}: V \to H_0$ such that $\hat{\mu}(\lambda_0) = \mu_0$, $\hat{u}(\lambda_0) = u_0$ and that $(\lambda,\mu,u) \in V \times W \times H_0$ satisfies (4.8) and $\|u\| = \|u_0\|$ if and only if $\mu = \hat{\mu}(\lambda)$ and either $u = \hat{u}(\lambda)$ or $u = -\hat{u}(\lambda)$. In particular, it follows by using the equivalence of (4.1) and (4.8) for $\lambda \approx \lambda_0$ and $u \approx u_0$ mentioned above that $(\lambda, \hat{\mu}(\lambda), \hat{u}(\lambda))$ satisfies the inequality (4.1) for all $\lambda \in V$ if V is small enough.

It remains to show that $(\lambda,\mu,u) = (\lambda, \hat{\mu}(\lambda), \hat{u}(\lambda))$ are the only solutions to (4.1) in $V \times W \times H$ with $\|u\| = \|u_0\|$. Let (λ_j, μ_j, u_j) be a sequence of solutions to (4.1) with

$$\|u_j\| = \|u_0\| \quad \text{for all } j \tag{4.9}$$

and with $\mu_j \to \mu_0$ and $\lambda_j \to \lambda_0$ for $j \to \infty$. It is sufficient to show that any such sequence satisfies $\mu_j = \hat{\mu}(\lambda_j)$ and $u_j = \hat{u}(\lambda_j)$ for large j. First, suppose $u_j \to u_0$ for $j \to \infty$. In that case, because of (4.3), (4.4) and Lemma 2.2 (for $F(\lambda,u) := (1/\mu_0)L(\lambda)u$ again), (λ_j, μ_j, u_j) is a solution to (4.8) for large j. In particular, $u_j \in H_0$ for large j, and by using the properties of the functions $\hat{\mu}(\lambda), \hat{u}(\lambda)$ stated above it follows that $\mu_j = \hat{\mu}(\lambda_j)$, $u_j = \hat{u}(\lambda_j)$. Now suppose that the sequence (u_j) does not converge to u_0. Then there exist an $\varepsilon > 0$ and a subsequence (u_{j_ℓ}) such that

$$\|u_{j_\ell} - u_0\| \geqslant \varepsilon \quad \text{for all } \ell. \tag{4.10}$$

Because of (4.9) we can assume that the subsequence (u_{j_ℓ}) converges weakly in H. Hence, the compactness of $L(\lambda_0)$ and the equalities

$$\mu_{j_\ell} u_{j_\ell} = P_K\big(L(\lambda_{j_\ell})u_{j_\ell}\big) = P_K\big(L(\lambda_0)u_{j_\ell} + O\big(\|\lambda_{j_\ell} - \lambda_0\|\big)\big)$$

imply that the subsequence (u_{j_ℓ}) converges strongly to a certain $u_* \in H$. Thus, (λ_0, μ_0, u_*) is a solution to (4.1) with $\|u_*\| = \|u_0\|$, and the assumption (4.7) yields $u_* = u_0$. This contradicts (4.10). □

Remark 4.1. Let a solution (λ_0, μ_0, u_0) to (4.1) and $c > 0$ be such that $\mu_0 > 0$ and that (4.2)–(4.4) hold. Then $u_0 \in \ker(\mu_0 I - PL(\lambda_0))$ (see the proof of Theorem 4.1). In particular, μ_0 is an eigenvalue of $PL(\lambda_0)$. This eigenvalue is geometrically simple if and only if (4.5) holds, and it is (algebraically) simple if and only if (4.5) and (4.6) hold. In particular, it is algebraically simple if (4.5) holds and if $L(\lambda_0)$ is symmetric. Let us notice that the simplicity assertions above are valid independently of whether $PL(\lambda_0)$ is considered as an operator on H_0 or on H. In particular, (4.6) holds if and only if $u_0 \notin (\mu_0 I - PL(\lambda_0))H$.

Remark 4.2. Let the assumptions of Theorem 4.1 be fulfilled. Then μ_0 is an (algebraically) simple eigenvalue of $PL(\lambda_0)$ (see the remark above), and [2, Section 14.2 and Theorem 14.3.1] imply the following: $\hat{\mu}(\lambda)$ is a simple eigenvalue of $PL(\lambda)$ and

$$\bigl\{\mu \approx \mu_0\colon \ \mu \in \operatorname{spec} PL(\lambda)\bigr\} = \bigl\{\hat{\mu}(\lambda)\bigr\} \quad \text{for all } \lambda \approx \lambda_0.$$

In particular, for all $\lambda \approx \lambda_0$ and $\mu \approx \mu_0$ with $\mu \neq \hat{\mu}(\lambda)$ we have that $\mu I - PL(\lambda)$ is an isomorphism on H as well as on H_0.

Remark 4.3. Suppose the assumptions of Theorem 4.1 to be satisfied. Then

$$\hat{\mu}'(\lambda_0)\lambda = \bigl\langle\bigl(L'(\lambda_0)\lambda\bigr)u_0, u_0^*\bigr\rangle \quad \text{for any } \lambda \in \Lambda, \tag{4.11}$$

where $u_0^* \in H_0$ is the unique element with $\langle u_0, u_0^*\rangle = 1$ and $\mu_0 u_0^* = PL(\lambda_0)^* u_0^*$. Here $L(\lambda_0)^* \in \mathcal{L}(H)$ is the adjoint operator to $L(\lambda_0)$. Indeed, μ_0 is a simple eigenvalue not only of the operator $PL(\lambda_0) : H_0 \to H_0$, but also of its adjoint operator, which is the restriction to H_0 of $PL(\lambda_0)^*$. Hence, there exists a $u_0^* \in H_0$ with $\mu_0 u_0^* = PL(\lambda_0)^* u_0^*$ and $\langle u_0, u_0^*\rangle = 1$. Differentiating the identity $\hat{\mu}(\lambda)\hat{u}(\lambda) = PL(\lambda)\hat{u}(\lambda)$ in $\lambda = \lambda_0$ and taking the scalar product with u_0^*, we get (4.11).

Corollary 4.1. *Let* (μ_0, λ_0, u_0) *satisfy* (4.1) *with* $\mu_0 > 0$ *and* $\mathcal{A} = \mathcal{A}_0 = \{1\}$. *Assume that* $L(\lambda_0)$ *is compact,*

there is no nontrivial solution to the equation

$$\mu_0 u = L(\lambda_0)u, \tag{4.12}$$

and (4.6) *is fulfilled. Then the assertion of Theorem* 4.1 *holds.*

Proof. We will show that the conditions (4.2)–(4.5) and (4.7) are automatically fulfilled if $\mathcal{A} = \mathcal{A}_0 = \{1\}$ and (4.12) holds.

If (4.2) were not true then we would have $\langle u_0, v_1\rangle > 0$, that means $u_0 \in \operatorname{int} K$. Hence, $\varphi = u_0 \pm \varepsilon\tilde{\varphi} \in K$ would hold for any $\tilde{\varphi} \in H$ with some $\varepsilon > 0$ small enough and (4.1) with $\mu = \mu_0$, $\lambda = \lambda_0$, $u = u_0$ would imply that u_0 satisfies the equation from (4.12), which is the contradiction.

The condition (4.3) means no restriction because $\mathcal{A} \setminus \mathcal{A}_0 = \emptyset$.

The condition (4.4) is equivalent to

$$\bigl\langle L(\lambda_0)u_0, v_1\bigr\rangle < 0 \tag{4.13}$$

by virtue of Lemma 2.3 (for $F(\lambda, u) := (1/\mu_0)L(\lambda)u$). If this condition were not true then we would get $L(\lambda_0)u_0 \in K$ and this together with (4.1) would give $\mu_0 u_0 - L(\lambda_0)u_0 = \mu_0 u_0 - P_K(L(\lambda_0)u_0) = 0$, which contradicts (4.12) (cf. Remark 2.3).

We have $u_0 \in \ker(\mu_0 I - PL(\lambda_0))$ by Remark 4.1. Let $v_0 \in \ker(\mu_0 I - PL(\lambda_0))$. Set $u := \langle L(\lambda_0)u_0, v_1\rangle v_0 - \langle L(\lambda_0)v_0, v_1\rangle u_0$. Then $\langle L(\lambda_0)u, v_1\rangle = 0$ and, hence, $PL(\lambda_0)u = L(\lambda_0)u$. Therefore, it follows from $u \in \ker(\mu_0 I - PL(\lambda_0))$ that $u \in \ker(\mu_0 I - L(\lambda_0))$, and (4.12) implies $u = 0$. That means $v_0 \in \operatorname{span}\{u_0\}$ by (4.13), and (4.5) is proved.

Let (μ_0, λ_0, u) be a solution to (4.1), $\|u\| = \|u_0\|$. Notice that

$$\big\langle L(\lambda_0)u, v_1\big\rangle < 0 \tag{4.14}$$

because otherwise we would have $L(\lambda_0)u \in K$ and this together with (4.1) would give $\mu_0 u - L(\lambda_0)u = \mu_0 u - P_K L(\lambda_0)u = 0$, which is excluded by (4.12). Moreover, $\langle u, v_1\rangle = 0$ because otherwise we would have $u \in \operatorname{int} K$ and u would be a nontrivial solution to the equation from (4.12) again (cf. the proof of (4.2) above). Set $\tilde{u} := \langle L(\lambda_0)u_0, v_1\rangle u - \langle L(\lambda_0)u, v_1\rangle u_0$. Then

$$\big\langle L(\lambda_0)\tilde{u}, v_1\big\rangle = 0, \qquad \langle \tilde{u}, v_1\rangle = 0. \tag{4.15}$$

For any $\varphi_0 \in H_0$ we have $u \pm \varphi_0 \in K$, $u_0 \pm \varphi_0 \in K$ and it follows from (4.1) (successively for u and u_0) that

$$\big\langle \mu_0 u - L(\lambda_0)u, \varphi_0\big\rangle = 0, \quad \big\langle \mu_0 u_0 - L(\lambda_0)u_0, \varphi_0\big\rangle = 0 \quad \text{for all } \varphi_0 \in H_0.$$

Hence, we get also

$$\big\langle \mu_0 \tilde{u} - L(\lambda_0)\tilde{u}, \varphi_0\big\rangle = 0 \quad \text{for all } \varphi_0 \in H_0,$$

and this together with (4.15) and the fact that $H = H_0 \oplus \operatorname{span}\{v_1\}$ gives $\mu_0 \tilde{u} - L(\lambda_0)\tilde{u} = 0$. Thus, we obtain from (4.12) that $\|\tilde{u}\| = 0$, that means

$$u = \frac{\langle L(\lambda_0)u, v_1\rangle}{\langle L(\lambda_0)u_0, v_1\rangle} u_0$$

(see also (4.13)). Since $\|u\| = \|u_0\|$ and (4.13), (4.14) hold, we get $u = u_0$ and (4.7) is proved.

Now, it is sufficient to use Theorem 4.1. □

Let us conclude this section by showing that Theorem 4.1 can be applied also to parameter depending eigenvalue problems of the type

$$\begin{aligned}&\lambda \in \Lambda,\ \nu \in \mathbb{R},\ u \in K: \quad \big\langle u - L(\lambda)u + \nu A(\lambda)u, \varphi - u\big\rangle \geqslant 0\\ &\quad \text{for all } \varphi \in K.\end{aligned} \tag{4.16}$$

Here $A : \Lambda \to \mathcal{L}(H)$ is a C^k-map such that $A(\lambda_0)$ is compact. The problem is to continue a solution (λ_0, ν_0, u_0) for $\lambda \approx \lambda_0$. In order to be able to use Theorem 4.1, we will consider the following problem with a new parameter

$\tilde{\lambda} = (\lambda, \nu) \in \tilde{\Lambda} := \Lambda \times \mathbb{R}$ and an eigenvalue parameter μ:

$$(\lambda, \nu) \in \tilde{\Lambda},\ \mu \in \mathbb{R},\ u \in K: \quad \left\langle \mu u - \frac{1}{\nu} L(\lambda)u + A(\lambda)u, \varphi - u \right\rangle \geqslant 0 \quad \text{for all } \varphi \in K. \tag{4.17}$$

This problem together with the equation

$$\nu = \frac{1}{\mu} \tag{4.18}$$

is equivalent to (4.16) for $\nu > 0$.

Let (λ_0, ν_0, u_0) be a solution to (4.16) with $\nu_0 > 0$, (4.2), (4.3),

$$\langle L(\lambda_0)u_0 - \nu_0 A(\lambda_0)u_0, (I-P)\varphi\rangle \leqslant -c\|(I-P)\varphi\| \quad \text{for all } \varphi \in K, \tag{4.19}$$

$$\dim\ker\bigl(I - P\bigl(L(\lambda_0) - \nu_0 A(\lambda_0)\bigr)\bigr) = 1, \tag{4.20}$$

$$u_0 \notin \bigl(I - P\bigl(L(\lambda_0) - \nu_0 A(\lambda_0)\bigr)\bigr) H_0, \tag{4.21}$$

and let the following condition be satisfied:

$$\text{if } (\lambda_0, \nu_0, u) \text{ is a solution to (4.16) with } \|u\| = \|u_0\|, \text{ then } \quad u = u_0. \tag{4.22}$$

Theorem 4.1 can be used for the mapping $\widetilde{L} : \widetilde{\Lambda} \to \mathcal{L}(H)$ defined by

$$\widetilde{L}(\tilde{\lambda})u := \widetilde{L}(\lambda, \nu)u = \begin{cases} \frac{1}{\nu} L(\lambda)u - A(\lambda)u & \text{for } \tilde{\lambda} = (\lambda, \nu),\ \nu \geqslant \frac{\nu_0}{2}, \\ \frac{2}{\nu_0} L(\lambda)u - A(\lambda)u & \text{for } \tilde{\lambda} = (\lambda, \nu),\ \nu < \frac{\nu_0}{2}, \end{cases}$$

and for $\mu_0 = 1/\nu_0$, $\tilde{\lambda}_0 = (\lambda_0, \nu_0)$. (Note that we work in a neighbourhood of ν_0 and in fact $\widetilde{L}(\lambda, \nu)$ for $\nu < \nu_0/2$ does not play any role.) It follows that (4.17) can be uniquely solved for $(\lambda, \nu) \approx (\lambda_0, \nu_0)$ in terms of $\mu \approx 1/\nu_0$ and $\|u\| = \|u_0\|$: $\mu = \hat{\mu}(\lambda, \nu)$, $u = \hat{u}(\lambda, \nu)$ with $\hat{\mu}(\lambda_0, \nu_0) = 1/\nu_0$ and $\hat{u}(\lambda_0, \nu_0) = u_0$. Inserting this solution into (4.18), we have to solve

$$G(\lambda, \nu) := \hat{\mu}(\lambda, \nu) - \frac{1}{\nu} = 0 \tag{4.23}$$

for $\lambda \approx \lambda_0$ in terms of $\nu \approx \nu_0$. Analogously to the proof of Theorem 4.1 we get

$$\hat{\mu}(\lambda, \nu)\hat{u}(\lambda, \nu) - P\left(\frac{1}{\nu} L(\lambda)\hat{u}(\lambda, \nu) - A(\lambda)\hat{u}(\lambda, \nu)\right) = 0.$$

Differentiating with respect to ν and multiplying by u_0^* in the solution $\lambda = \lambda_0$, $\nu = \nu_0$ we obtain

$$\frac{\partial G}{\partial \nu}(\lambda_0, \nu_0) = \frac{\partial \hat{\mu}}{\partial \nu}(\lambda_0, \nu_0) + \frac{1}{\nu_0^2} = \frac{1}{\nu_0^2}\bigl(1 - \langle L(\lambda_0)u_0, u_0^*\rangle\bigr)$$

(cf. (4.11)), where $u_0^* \in H_0$ is defined by $\langle u_0, u_0^* \rangle = 1$ and $\mu_0 u_0^* = P((1/\nu_0) \times L(\lambda_0)^* - A(\lambda_0)^*)u_0^*$. Hence, if

$$\langle L(\lambda_0)u_0, u_0^* \rangle \neq 1, \tag{4.24}$$

then (4.23) is locally uniquely solvable, and we get the following results:

Theorem 4.2. *Let* (λ_0, ν_0, u_0) *be a solution to* (4.16) *with* $\nu_0 > 0$, (4.2), (4.3), (4.19)–(4.22) *and* (4.24). *Moreover, let* $L(\lambda_0)$ *and* $A(\lambda_0)$ *be compact.*

Then there exist neighbourhoods $V \subseteq \Lambda$ *of* λ_0 *and* $W \subseteq \mathbb{R}$ *of* ν_0 *and* C^k*-maps* $\hat{\nu} : V \to \mathbb{R}$ *and* $\hat{u} : V \to H_0$ *such that* $(\lambda, \nu, u) \in V \times W \times H$ *satisfies* (4.16) *with* $\|u\| = \|u_0\|$ *if and only if* $\nu = \hat{\nu}(\lambda)$ *and* $u = \hat{u}(\lambda)$. *In particular,* $\nu_0 = \hat{\nu}(\lambda_0)$ *and* $u_0 = \hat{u}(\lambda_0)$.

Corollary 4.2. *Let* (λ_0, ν_0, u_0) *be a solution to* (4.16) *with* $\nu_0 > 0$ *and* $\mathcal{A} = \mathcal{A}_0 = \{1\}$. *Assume that* $L(\lambda_0)$ *and* $A(\lambda_0)$ *are compact,*

there is no nontrivial solution to the equation

$$u = \big(L(\lambda_0) - \nu_0 A(\lambda_0)\big)u, \tag{4.25}$$

and (4.21) *and* (4.24) *are fulfilled. Then the assertion of Theorem* 4.2 *holds.*

5. Applications to an elliptic eigenvalue problem with unilateral integral boundary conditions

5.1. Application of Corollary 4.2

As an example of a problem with an infinite-dimensional control parameter, we will consider the eigenvalue problem

$$\Delta u(x) + \lambda(x)u(x) = \nu u(x) \quad \text{in } \Omega, \tag{5.1}$$

$$u = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_U), \tag{5.2}$$

$$\int_{\Gamma_U} u \, \mathrm{d}\Gamma \geqslant 0, \qquad \frac{\partial u}{\partial n} \geqslant 0, \qquad \frac{\partial u}{\partial n} \text{ is constant,}$$

$$\int_{\Gamma_U} u \, \mathrm{d}\Gamma \cdot \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_U. \tag{5.3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ with a Lipschitzian boundary $\partial\Omega$, Γ_D and Γ_U are relatively open subsets of this boundary with

$$\Gamma_U \cap \Gamma_D = \emptyset, \quad \operatorname{meas} \Gamma_D > 0$$

(the $(n-1)$-dimensional Lebesgue measure), the coefficient $\lambda \in L^\infty(\Omega)$ is understood as the control parameter, and $\nu \in \mathbb{R}$ is the eigenvalue parameter.

Let us introduce a weak formulation for (5.1)–(5.3) which fits into the framework of Section 4. The vector space

$$H := \bigl\{u \in W^{1,2}(\Omega)\colon\ u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\bigr\}$$

is a Hilbert space with the inner product

$$\langle u, v\rangle := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{for any } u, v \in H.$$

The corresponding norm (denoted by $\|\cdot\|$) is equivalent to the usual Sobolev norm on H. We introduce the closed convex cone K (in fact a halfspace) and the corresponding subspace H_0 by

$$K := \left\{u \in H\colon \int_{\Gamma_U} u \,\mathrm{d}\Gamma \geqslant 0\right\}, \qquad H_0 := \left\{u \in H\colon \int_{\Gamma_U} u \,\mathrm{d}\Gamma = 0\right\}.$$

Define operators $L : L^\infty(\Omega) \to \mathcal{L}(H)$ and $A \in \mathcal{L}(H)$ by

$$\bigl\langle L(\lambda)u, \varphi\bigr\rangle = \int_\Omega \lambda(x) u\varphi \,\mathrm{d}x, \qquad \langle Au, \varphi\rangle = \int_\Omega u\varphi \,\mathrm{d}x$$
$$\text{for all } \lambda \in L^\infty(\Omega) \text{ and } u, \varphi \in H.$$

Obviously, L is smooth (because it is linear and continuous), and $L(\lambda)$ and A are compact operators on H.

We will say that (λ, ν, u) satisfies (5.1)–(5.3) in the weak sense if the variational inequality (4.16) is fulfilled with the cone and operators just introduced, i.e., if

$$\begin{aligned} &\lambda \in L^\infty(\Omega),\ \nu \in \mathbb{R},\ u \in K\colon \\ &\quad \int_\Omega \nabla u \cdot \nabla(\varphi - u) - \bigl(\lambda(x) - \nu\bigr) u(\varphi - u) \,\mathrm{d}x \geqslant 0 \quad \text{for all } \varphi \in K. \end{aligned} \tag{5.4}$$

Analogously, a weak solution of Eq. (5.1) with classical boundary conditions

$$u = 0 \quad \text{on } \Gamma_D, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \setminus \Gamma_D, \tag{5.5}$$

and with fixed $\lambda \in L^\infty(\Omega)$, $\nu \in \mathbb{R}$, is a function $u \in H$ such that

$$\int_\Omega \nabla u \cdot \nabla\varphi - \bigl(\lambda(x) - \nu\bigr) u\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in H. \tag{5.6}$$

By an eigenvalue of (5.1), (5.5) (with a given $\lambda \in L^\infty(\Omega)$) we mean a number $\nu \in \mathbb{R}$ such that there exists a nontrivial weak solution to (5.1), (5.5).

Theorem 5.1. *Let* (λ_0, ν_0, u_0) *with* $\nu_0 > 0$ *satisfy* (5.1)–(5.3) *in the weak sense. Suppose that*

$$\nu_0 \text{ is not an eigenvalue of the problem (5.1), (5.5) with } \lambda = \lambda_0, \tag{5.7}$$

$$\int_\Omega \lambda_0(x) u_0^2 \,\mathrm{d}x \neq \int_\Omega |\nabla u_0|^2 \,\mathrm{d}x. \tag{5.8}$$

Then there exist neighbourhoods $V \subset L^\infty(\Omega)$ *of* λ_0, $W \subset \mathbb{R}$ *of* ν_0 *and* C^∞*-maps* $\hat{\nu} : V \to \mathbb{R}$ *and* $\hat{u} : V \to H_0$ *such that* $(\lambda, \nu, u) \in V \times W \times H$ *satisfies* (5.1)–(5.3) *in the weak sense with* $\|u\| = \|u_0\|$ *if and only if* $\nu = \hat{\nu}(\lambda)$, $u = \hat{u}(\lambda)$. *In particular,* $\nu_0 = \hat{\nu}(\lambda_0)$ *and* $u_0 = \hat{u}(\lambda_0)$.

Proof. In order to use the notation of Section 4 let us set $\mathcal{A} = \mathcal{A}_0 := \{1\}$ and let $v_1 \in H$ be such that $\langle u, v_1 \rangle = \int_{\Gamma_U} u \,\mathrm{d}\Gamma$ for all $u \in H$. Then $K = \{u \in H\colon \langle u, v_1 \rangle \geqslant 0\}$ and $H_0 = \{u \in H\colon \langle u, v_1 \rangle = 0\}$. We will verify the assumptions of Corollary 4.2. The condition (5.7) is equivalent to (4.25). Let us notice that $u_0 \in H_0$. Indeed, otherwise we would have $\int_{\Gamma_U} u_0 \,\mathrm{d}\Gamma > 0$, and it would follow from (5.4) that u_0 is a nontrivial weak solution to (5.1), (5.5) with $\lambda = \lambda_0$, $\nu = \nu_0$, which would contradict the assumption (5.7). By virtue of Lemma 2.2 (for $F(\lambda, u) := L(\lambda)u - \nu_0 A u$) we have $u_0 \in \ker(I - P(L(\lambda_0) - \nu_0 A))$. The operators $L(\lambda)$ and A are symmetric and therefore (4.21) is fulfilled. The element u_0^* from the condition (4.24) satisfies

$$u_0^* = \frac{u_0}{\|u_0\|^2}, \qquad \left\langle L(\lambda_0)u_0, u_0^* \right\rangle = \frac{\langle L(\lambda_0)u_0, u_0 \rangle}{\|u_0\|^2},$$

and the condition (4.24) follows from (5.8). Now, our assertion is a consequence of Corollary 4.2. □

Remark 5.1. The solutions $(\lambda, \nu, u) = (\lambda, \hat{\nu}(\lambda), \hat{u}(\lambda))$ produced by Theorem 5.1 satisfy the conditions

$$\int_{\Gamma_U} u \,\mathrm{d}\Gamma = 0,$$

$$\frac{\partial u}{\partial n} > 0, \qquad \frac{\partial u}{\partial n} \text{ is constant on } \Gamma_U.$$

The precise sense of the last condition will be seen from Remark 5.2 below. The assertion will be contained in a more general form in Theorem 5.3 proved in the second part of this section.

Remark 5.2. If $u \in H$ is a function such that $\Delta u \in L^2(\Omega)$, then the normal derivative $\partial u / \partial n$ can be defined as a linear bounded functional on the space H by

$$\left[\frac{\partial u}{\partial n}, \varphi\right] = \int_\Omega \Delta u \varphi + \nabla u \nabla \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H.$$

Here $[\cdot\,,\cdot]$ is the dual pairing. If u is sufficiently smooth up to the boundary then, of course,

$$\left[\frac{\partial u}{\partial n},\varphi\right]=\int\limits_{\partial\Omega}\frac{\partial u}{\partial n}\varphi\,\mathrm{d}\Gamma\quad\text{for all }\varphi\in H,$$

where $\partial u/\partial n$ in the integral on the right hand side is the classical derivative of u with respect to the outer normal to $\partial\Omega$ (that means the classical Green formula holds). As usual, given a relatively open subset Γ of $\partial\Omega$, we will say that

- $\partial u/\partial n=0$ on Γ if $[\partial u/\partial n,\varphi]=0$ for all $\varphi\in H$ with $\varphi=0$ on $\partial\Omega\setminus\Gamma$,
- $\partial u/\partial n\geqslant 0$ on Γ if $[\partial u/\partial n,\varphi]\geqslant 0$ for all $\varphi\in H$ with $\varphi=0$ on $\partial\Omega\setminus\Gamma$ and $\varphi\geqslant 0$ on Γ,
- $\partial u/\partial n>0$ on Γ if $[\partial u/\partial n,\varphi]>0$ for all $\varphi\in H$ with $\varphi=0$ on $\partial\Omega\setminus\Gamma$ and $\varphi\geqslant 0$ on Γ and not $\varphi=0$ on Γ,
- $\partial u/\partial n$ is constant on Γ if there is $c\in\mathbb{R}$ such that $[\partial u/\partial n,\varphi]=c\int_\Gamma\varphi\,\mathrm{d}\Gamma$ for all $\varphi\in H$ with $\varphi=0$ on $\partial\Omega\setminus\Gamma$,

where all equalities and inequalities for φ on parts of $\partial\Omega$ are understood in the sense of traces.

5.2. *Application of Theorem 4.1*

We can consider also problems with a finite number of constraints. Let Γ_j, $j=1,\dots,n$, be open (in $\partial\Omega$) subsets of $\partial\Omega\setminus\Gamma_D$ and such that the sets $\overline{\Gamma}_j$ (the closures) are pairwise disjoint. In order to be able to use Theorem 4.1, let us introduce the notation $\mathcal{A}=\{1,\dots,n\}$ and fix a subset $\mathcal{A}_0=\{\alpha_1,\dots,\alpha_m\}$ of $\mathcal{A}$.

We say that (λ,ν,u) satisfies in the weak sense the equation (5.1),

$$u=0\quad\text{on }\Gamma_D,\qquad \frac{\partial u}{\partial n}=0\quad\text{on }\partial\Omega\Bigg\backslash\left(\Gamma_D\cup\bigcup_{j=1}^{n}\Gamma_j\right),\tag{5.9}$$

$$\int\limits_{\Gamma_j}u\,\mathrm{d}\Gamma\geqslant 0,\qquad \frac{\partial u}{\partial n}\geqslant 0,\qquad \frac{\partial u}{\partial n}\text{ is constant,}$$

$$\int\limits_{\Gamma_j}u\,\mathrm{d}\Gamma\cdot\frac{\partial u}{\partial n}=0\quad\text{on }\Gamma_j,\ j=1,\dots,n,\tag{5.10}$$

if the variational inequality (4.16) is fulfilled (that means (5.4) holds) with the cone

$$K:=\left\{u\in H\colon \int\limits_{\Gamma_j}u\,\mathrm{d}\Gamma\geqslant 0,\ j=1,\dots,n\right\}.$$

Observation 5.2. *If* (λ, ν, u) *satisfies* (5.1), (5.9), (5.10) *in the weak sense then* $\Delta u \in L^2(\Omega)$, *Eq.* (5.1) *holds a.e. in* Ω *and the boundary conditions* (5.9), (5.10) *are fulfilled with* $\partial u/\partial n$ *understood in the sense of Remark* 5.2. *Moreover, there exist* $c_j \geqslant 0$, $j = 1, \dots, n$, *such that*

$$\left[\frac{\partial u}{\partial n}, \varphi\right] = \sum_{j=1}^{n} c_j \int_{\Gamma_j} \varphi \, \mathrm{d}\Gamma \quad \textit{for all } \varphi \in H. \tag{5.11}$$

Proof. If (λ, ν, u) satisfies (5.1), (5.9), (5.10) in the weak sense then we obtain the Eq. (5.1) a.e. in Ω by taking all test functions $\varphi := u \pm \psi$, $\psi \in \mathcal{D}(\Omega)$ in (5.4). We get $\Delta u \in L^2(\Omega)$ from Eq. (5.1) and therefore $\partial u/\partial n$ is defined as a functional (see Remark 5.2). Multiplying (5.1) by $\varphi - u$ and integrating we get, by using Remark 5.2, that

$$\int_{\Omega} \nabla u \cdot \nabla(\varphi - u) - \big(\lambda(x) - \nu\big) u(\varphi - u) \, \mathrm{d}x - \left[\frac{\partial u}{\partial n}, \varphi - u\right] = 0$$
$$\text{for all } \varphi \in K.$$

It follows from the last equality by using (5.4) that

$$\left[\frac{\partial u}{\partial n}, \varphi - u\right] \geqslant 0 \quad \text{for all } \varphi \in K. \tag{5.12}$$

The first condition in (5.10) is obvious because $u \in K$. Furthermore, $\varphi := u + \psi \in K$ for any $\psi \in H$, $\psi = 0$ on $\partial\Omega \setminus \Gamma_j$, $\psi \geqslant 0$ on Γ_j, and we obtain from (5.12) that $[\partial u/\partial n, \psi] \geqslant 0$ for all such ψ, that means $\partial u/\partial n \geqslant 0$ on Γ_j, $j = 1, \dots, n$.

Suppose that $\int_{\Gamma_j} u \, \mathrm{d}\Gamma > 0$ for some j. Then for any $\psi \in H$, $\psi = 0$ on $\partial\Omega \setminus \Gamma_j$, there exists $\varepsilon > 0$ such that $\varphi := u \pm \varepsilon\psi \in K$. Hence, we get from (5.12) that

$$\left[\frac{\partial u}{\partial n}, \psi\right] = 0 \quad \text{for any } \psi \in H,\ \psi = 0 \text{ on } \partial\Omega \setminus \Gamma_j,$$

that means $\partial u/\partial n = 0$ in Γ_j for j under consideration and the last condition in (5.10) follows.

Now let us show that for any $j = 1, \dots, n$ there is $c_j \geqslant 0$ such that

$$\left[\frac{\partial u}{\partial n}, \varphi\right] = c_j \int_{\Gamma_j} \varphi \, \mathrm{d}\Gamma$$
$$\text{for all } \varphi \in H,\ \varphi = 0 \text{ on } \Gamma_k \text{ for } k = 1, \dots, n,\ k \neq j. \tag{5.13}$$

If this were not true for some j then $c_j^{(1)}, c_j^{(2)} \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in H$ would exist such that

$$\left[\frac{\partial u}{\partial n}, \varphi_i\right] = c_j^{(i)} \int_{\Gamma_j} \varphi_i \, \mathrm{d}\Gamma, \quad i = 1, 2,$$

$$\varphi_i = 0 \text{ on } \Gamma_k \text{ for } k = 1, \ldots, n, \; k \neq j,$$

$$\int_{\Gamma_j} \varphi_1 \, \mathrm{d}\Gamma = \int_{\Gamma_j} \varphi_2 \, \mathrm{d}\Gamma \neq 0, \quad c_j^{(1)} \neq c_j^{(2)}.$$

Therefore,

$$\left[\frac{\partial u}{\partial n}, \varphi_1 - \varphi_2\right] = \left(c_j^{(1)} - c_j^{(2)}\right) \int_{\Gamma_j} \varphi_1 \, \mathrm{d}\Gamma \neq 0.$$

Simultaneously $[\partial u/\partial n, \varphi_1 - \varphi_2] = 0$, because we can choose $\varphi := u \pm \widetilde{\varphi} \in K$ with $\widetilde{\varphi} = \varphi_1 - \varphi_2$ in (5.12). This is a contradiction and (5.13) is proved. In particular, the second condition from (5.9) and the constantness of $\partial u/\partial n$ on Γ_j, $j = 1, \ldots, n$, from (5.10) follow.

Since there is a positive distance between $\overline{\Gamma}_j$ and $\overline{\Gamma}_k$ for $j \neq k$, $j, k = 1, \ldots, n$, there exists a system of open sets $V_j \subset \mathbb{R}^N$, $j = 0, 1, \ldots, n$, covering $\overline{\Omega}$ such that $\overline{\Gamma}_j \subset V_j$, $V_0 \cap \overline{\Gamma}_j = \emptyset$ for $j = 1, \ldots, n$, $\overline{\Gamma}_j \cap V_k = \emptyset$ for $j \neq k$. Let β_j, $j = 0, 1, \ldots, n$, be a smooth partition of unity subordinated to this covering. Then for any $\varphi \in H$ the functions $\varphi_j := \beta_j \varphi$ and $\varphi_0 := (1 - \sum_{j=1}^n \beta_j)\varphi$ belong to H, and for $j, k = 1, \ldots, n$ they satisfy the following properties: $\varphi_j = \varphi$ on Γ_j, $\varphi_j = 0$ on Γ_k for $k \neq j$ and $\varphi_0 = 0$ on Γ_k for all k. By using (5.10) and (5.13) for φ_j we obtain

$$\left[\frac{\partial u}{\partial n}, \varphi\right] = \left[\frac{\partial u}{\partial n}, \varphi_0 + \sum_{j=1}^n \varphi_j\right] = \sum_{j=1}^n c_j \int_{\Gamma_j} \varphi \, \mathrm{d}\Gamma \quad \text{for all } \varphi \in H,$$

and the proof is complete. □

There exist functions $v_1, \ldots, v_n \in H$ such that

$$\langle \varphi, v_\alpha \rangle = \int_{\Gamma_\alpha} \varphi \, \mathrm{d}\Gamma \quad \text{for all } \varphi \in H, \; \alpha \in \mathcal{A}, \tag{5.14}$$

and functions $v^*_{\alpha_1}, \ldots, v^*_{\alpha_m} \in \operatorname{span}\{v_{\alpha_1}, \ldots, v_{\alpha_m}\}$ such that

$$\left\langle v_\alpha, v^*_\beta \right\rangle = \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta \in \mathcal{A}_0. \tag{5.15}$$

Set

$$H_0 := \left(\operatorname{span}\{v_\alpha \colon \alpha \in \mathcal{A}_0\}\right)^\perp = \left\{u \in H \colon \int_{\Gamma_\alpha} u \, \mathrm{d}\Gamma = 0 \text{ for all } \alpha \in \mathcal{A}_0\right\}.$$

Finally, we introduce the notion of a weak solution to (5.1) with the conditions (5.9),

$$\int_{\Gamma_\alpha} u \,\mathrm{d}\Gamma = 0 \quad \text{for } \alpha \in \mathcal{A}_0, \tag{5.16}$$

$$\frac{\partial u}{\partial n} \text{ is constant on } \Gamma_\alpha \text{ for any } \alpha \in \mathcal{A}_0, \tag{5.17}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_\alpha \text{ for } \alpha \in \mathcal{A} \setminus \mathcal{A}_0 \tag{5.18}$$

as a function u satisfying

$$u \in H_0\colon \quad \int_\Omega \nabla u \cdot \nabla \varphi - \big(\lambda(x) - \nu\big) u\varphi \,\mathrm{d}x = 0$$
$$\text{for all } \varphi \in H_0. \tag{5.19}$$

As above one can show the following: If (λ, ν, u) satisfies (5.1), (5.9), (5.16)–(5.18) in the weak sense, then $\Delta u \in L^2(\Omega)$, Eq. (5.1) holds a.e. in Ω and the boundary conditions (5.9), (5.16)–(5.18) are fulfilled with $\partial u/\partial n$ understood in the sense of Remark 5.2.

Theorem 5.3. *Let* (λ_0, ν_0, u_0) *with* $\nu_0 > 0$ *satisfy* (5.1), (5.9), (5.10) *in the weak sense and let the conditions* (5.8), (5.16), (5.18),

$$\int_{\Gamma_\alpha} u \,\mathrm{d}\Gamma > 0 \quad \textit{for } \alpha \in \mathcal{A} \setminus \mathcal{A}_0, \tag{5.20}$$

$$\frac{\partial u}{\partial n} > 0, \qquad \frac{\partial u}{\partial n} \textit{ is constant on } \Gamma_\alpha \textit{ for any } \alpha \in \mathcal{A}_0 \tag{5.21}$$

be fulfilled for $u = u_0$. *Let us assume the following unicity conditions*:

> *If* (λ_0, ν_0, v_0) *satisfies* (5.1), (5.9), (5.16)–(5.18) *in the weak sense then there exists a constant* $c \in \mathbb{R}$ *such that* $v_0 = cu_0$. (5.22)
>
> *If* (λ_0, ν_0, v_0) *satisfies* (5.1), (5.9), (5.10) *in the weak sense then there exists a constant* $c \geqslant 0$ *such that* $v_0 = cu_0$. (5.23)

Then there exist neighbourhoods $V \subset L^\infty(\Omega)$ *of* λ_0 *and* $W \subset \mathbb{R}$ *of* ν_0 *and* C^∞*-maps* $\hat{\nu}: V \to \mathbb{R}$ *and* $\hat{u}: V \to H_0$ *such that* $(\lambda, \nu, u) \in V \times W \times H$ *satisfies* (5.1), (5.9), (5.10) *in the weak sense with* $\|u\| = \|u_0\|$ *if and only if* $\nu = \hat{\nu}(\lambda)$, $u = \hat{u}(\lambda)$. *In particular,* $\nu_0 = \hat{\nu}(\lambda_0)$ *and* $u_0 = \hat{u}(\lambda_0)$. *Moreover, for all* $u = \hat{u}(\lambda)$, $\lambda \in V$, *the conditions* (5.16), (5.20) *and* (5.21) *are fulfilled.*

Proof. Let us check the assumptions of Theorem 4.2. The condition (4.2) is equivalent to (5.16) and (4.3) is equivalent to (5.20).

Let us verify the condition (4.19). Let $\partial u_0/\partial n = c_\alpha$ on Γ_α (see (5.10)). For $v \in H$ we obtain by using Observation 5.2, Remark 5.2, (5.9), (5.18) and (5.11) (by observing that (5.18) implies $c_\alpha = 0$ for $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$) that

$$\begin{aligned}\big\langle L(\lambda_0)u_0 - \nu_0 A u_0, v\big\rangle &= \int_\Omega \big(\lambda_0(x) - \nu_0\big) u_0 v \,\mathrm{d}x \\ &= -\int_\Omega \Delta u_0 \cdot v \,\mathrm{d}x = \langle u_0, v\rangle - \left[\frac{\partial u_0}{\partial n}, v\right] \\ &= \langle u_0, v\rangle - \sum_{\alpha \in \mathcal{A}_0} c_\alpha \int_{\Gamma_\alpha} v \,\mathrm{d}\Gamma. \end{aligned} \tag{5.24}$$

Taking $v := v_\beta^*$ and using (5.14), (5.15) we get $\int_{\Gamma_\alpha} v_\beta^* \,\mathrm{d}\Gamma = \delta_{\alpha\beta}$. Therefore by using (5.16) for u_0 we obtain $\langle L(\lambda_0)u_0 - \nu_0 A u_0, v_\beta^*\rangle = -c_\beta$ for all $\beta \in \mathcal{A}_0$. Thus, by virtue of Lemma 2.3 (for $F(\lambda, u) := L(\lambda)u - \nu_0 A u$), the condition (4.19) follows from (5.21).

It is easy to see that if $v_0 - P(L(\lambda_0)v_0 - \nu_0 A v_0) = 0$ then v_0 is a weak solution to (5.1), (5.9), (5.16)–(5.18). Hence, the condition (4.20) follows from (5.22).

The condition (4.21) is automatically fulfilled owing to the symmetry of the operators $L(\lambda_0)$ and A (cf. Remark 4.1).

The condition (4.22) follows from (5.23).

Since $u = \hat{u}(\lambda) \in H_0$, the condition (5.16) is fulfilled for $\hat{u}(\lambda)$, $\lambda \in V$. The condition (5.20) is supposed for $u = u_0$ and it remains valid for $u = \hat{u}(\lambda)$ due to continuity. Also the condition (4.19) (already proved) remains valid if we replace (λ_0, ν_0, u_0) by $(\lambda, \hat{\nu}(\lambda), \hat{u}(\lambda))$. Analogously as above (cf. (5.24)), we can show that the condition (5.21) for $u = \hat{u}(\lambda)$ follows. □

Remark 5.3. We can replace Eq. (5.1) by

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \lambda(x)u = \nu u \quad \text{in } \Omega, \tag{5.25}$$

where $a_{ij} \in L^\infty(\Omega)$ are given functions satisfying the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2 \quad \text{for all } \xi = [\xi_1, \ldots, \xi_n] \in \mathbb{R}^n, \text{ a.a. } x \in \Omega.$$

We use again the space $H := \{u \in W^{1,2}(\Omega)\colon u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}$ but this time with the inner product

$$\langle u, v\rangle = \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x$$

generating an equivalent norm on H. We introduce the operators $L(\lambda)$, $A(\lambda)$ as in the case of Eq. (5.1) but by using the new inner product and use again the cone K corresponding to the boundary conditions (5.2), (5.3) or (5.9), (5.10) as above. Then the weak formulation of the problem (5.25) with (5.2), (5.3) or (5.9), (5.10), respectively, is the variational inequality (4.16). Analogues of Theorems 5.1 and 5.3 hold.

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